## Last time: CTMC's

- A (temporally homogeneous) CTMC is a process $(X_t)_{t\geq 0}$ taking values in discrete "state space" $S$, such that:

$$P\{X_{t+\tau}=j \mid X_t=i,\ X_s=i_s,\ s<t\} = P\{X_\tau=j \mid X_0=i\}$$

$$\forall i,j\in S \quad \tau, t\geq 0$$

Fact: dynamics are completely characterized by "rate matrix" $Q$

$$Q = \begin{bmatrix} -q_1 & q_{12} & q_{13} & \cdots \\ q_{21} & -q_2 & q_{23} & \cdots \\ \vdots & & \ddots & \end{bmatrix}$$

$Q$ matrix properties:

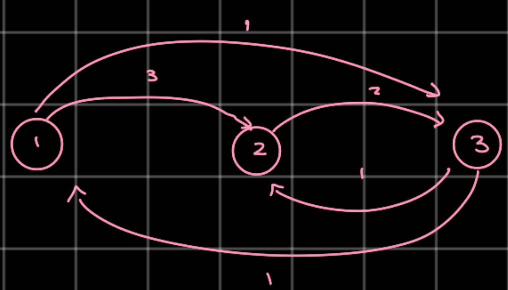
- off-diagonal entries are $\geq 0$
- rows sum to $0$
- $(-q_i)$ = transition rate from $i\to j$
- $q_{ij}$ = transition prob from $i\to j$
- $(P_{ij}) := \dfrac{q_{ij}}{q_i}$ = transition prob $i\to j$ jump chain (DTMC)

Recipe

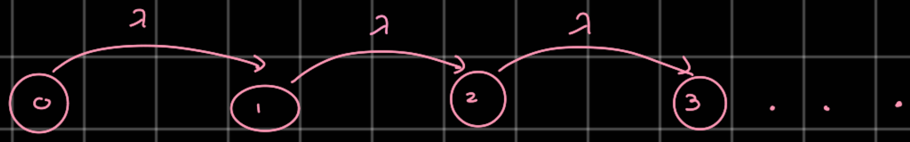
- sequence of states is determined by the jump chain
- Hold in state $i$ for $\text{Exp}(q_i)$ time before departing

ex:

$$Q = \begin{bmatrix} -4 & 3 & 1 \\ 0 & -2 & 2 \\ 1 & 1 & -2 \end{bmatrix}$$




Key Skill: take problem description & model it as a CTMC example PP($\lambda$)




$$q_n = q_{n,n+1} = \lambda \qquad \forall n \geq 0$$

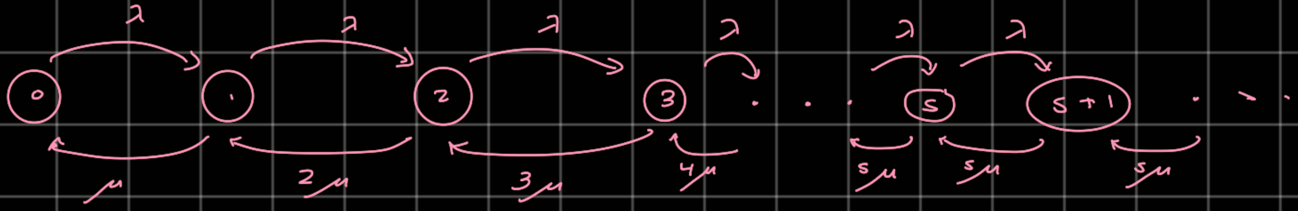
example : M / M / S Queue

memoryless arrivals

memoryless service time

# servers

- customers arrive to a store according to PP($\lambda$)
- $(X_t)$ = # customers in store at time $t \geq 0$
- if customer arrives and a server is available, they immediately enter service. If all servers are busy, they wait in line. Service times are iid Exp($\mu$)



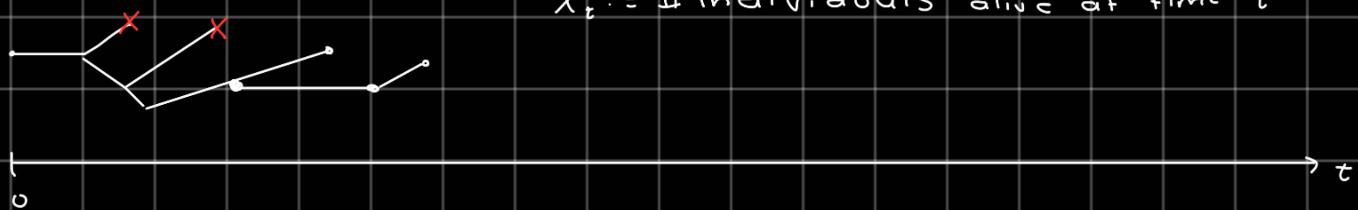

$$q_{n,n+1} = \lambda \qquad n \geq 0$$

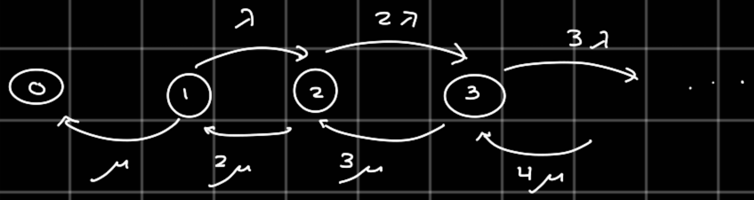
$$q_{n,n-1} = \begin{cases} n\mu & 0 \leq n \leq s \\ s\mu & n > s \end{cases}$$

example : Birth-Death Process

- Each individual gives birth at rate $\lambda$, individual of all others
- individuals have lifetimes $\overset{\sim}{\text{iid}}$ Exp($\mu$)

$X_t$ := # individuals alive at time $t$






if $\mu > \lambda$
↳ wp 1 you go extinct at some point
if $\mu < \lambda$
↳ you go extinct with some prob

$q_{n,n+1} = n\lambda$ $\qquad$ $q_{n,n-1} = n\mu$

Q/ What is jump chain?

$$P_{n,n+1} = \frac{q_{n,n+1}}{q_n} = \frac{q_{n,n+1}}{q_{n,n-1} + q_{n,n+1}} = \frac{\lambda}{\lambda+\mu}$$

$$P_{n,n-1} = \frac{\mu}{\lambda+\mu}$$

jump chain is a DT birth-death process

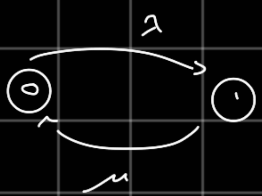
Just like in DTMCs, we want to be able to characterize long-term (steady state) behavior.

Subject to technical conditions (which we shamelessly ignore), a prob vector $\pi$ is a SD for CTMC with rate matrix $Q$ if

$$\pi Q = 0$$

- "rate in" = "rate out" at each state
- rate conservation principle
- BEs

Example:




$$Q = \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix}$$

↳ want to solve $\pi Q = 0$

$$\pi(0)\lambda = \pi(1)\lambda$$

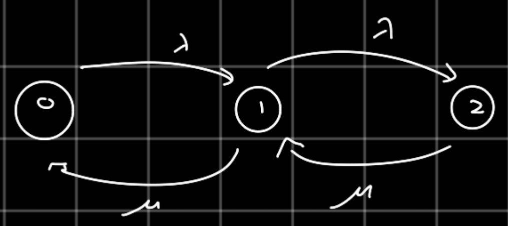
$$\Rightarrow \frac{\pi(0)}{\pi(1)} = \frac{\mu}{\lambda}$$

$$\pi(0) = \frac{\mu}{\lambda+\mu} \qquad \pi(1) = \frac{\lambda}{\lambda+\mu}$$

Ex: I have 2 sensors; one in use & one as a backup

- when in use, sensor lasts Exp($\mu$) time before breaking.
- backup goes into service, if available
- broken sensors respond at rate $\lambda$. shop can only fix one sensor at a time

$X_t$ = # working sensors at time $t=0$



Q/ What's long-run factor of time where both sensors are broken?

↳ Find $\pi$, answer is $\pi(0)$

$$Q = \begin{bmatrix} -\lambda & \lambda & 0 \\ \mu & (-\lambda+\mu) & \lambda \\ 0 & \mu & -\mu \end{bmatrix}$$

$$\lambda \pi(0) = \mu \pi(1) \Rightarrow \pi(1) = \frac{\lambda}{\mu}\pi(0)$$

$$\lambda \pi(1) = \mu \pi(2) \Rightarrow \pi(2) = \frac{\lambda}{\mu}\pi(1) = \left(\frac{\lambda}{\mu}\right)^2 \pi(0)$$

$$1 = \pi(0) + \pi(1) + \pi(2) = \pi(0)\left(1 + \frac{\lambda}{\mu} + \left(\frac{\lambda}{\mu}\right)^2\right)$$

$$\therefore \pi(0) = \left(1 + \frac{\lambda}{\mu} + \left(\frac{\lambda}{\mu}\right)^2\right)^{-1}$$

## Classification of States:

- classes of states in CTMC coincide with the classes in DTMC jump chain

- Chain is irreducible if there's only 1 class

- state $j$ is transient if, given $X_0 = j$ $(X_t)_{t\geq 0}$ revisits state $j$ finitely many times wp 1. State $j$ is recurrent otherwise
  - For $j$ recurrent, define

$$T_j = \min\{t \geq 0 : X_t = j \text{ and } \exists s < t \text{ s.t } X_s \neq j\}$$

$$= 1^{st} \text{ re-entry time into state } j$$

- $j$ positive recurrent if

$$\mathbb{E}[T_j \mid X_0 = j] < \infty$$

- $j$ null recurrent if

$$\mathbb{E}[T_j \mid X_0 = j] = \infty$$

- No concept of periodicity in CTMCs ü

**BIG THEOREM**: Define $P_{ij}^{t} := P\{X_t = j \mid X_0 = j\}$

For irreducible CTMC, exactly 1 of the following is true:

1) All states transient or all states null recurrent, No SD exists and

$$\lim_{t\to\infty} P_{ij}^{t} = 0 \quad \forall i,j$$

2) All states positive recurrent, unique SD exists, satisfies

$$\pi(j) = \frac{1}{q_j \, \mathbb{E}[T_j \mid X_0 = j]} = \lim_{t\to\infty} P_{ij}^{t} \quad \forall i,j \in S$$

Note: SD in CTMC is not generally the same as SD for jump chain, but they're related as follows:

$$\underbrace{\pi_j^{(jump)}}_{\text{SD for jump chain}} = \frac{\pi_j q_j}{\sum_i \pi_i q_i} \longleftarrow \text{works provided } \sum \pi_i q_i < \infty$$

Ex: $M/M/\infty$ Queue

$q_{n,n+1} = \lambda$ (arrivals $PP(\lambda)$)

$q_{n,n-1} = n\mu$ (service times $\overset{}{\underset{iid}{\sim}}$ $Exp(\mu)$)

infinite # of servers

Q1/ Long-term distribution of $X_t$?

$$\pi Q = 0$$

Guess that $\pi_n = \dfrac{e^{-\frac{\lambda}{\mu}} \left(\frac{\lambda}{\mu}\right)^n}{n!}$

$\pi \sim Pois\left(\frac{\lambda}{\mu}\right)$

i.e. by Big THM $X_t \xrightarrow{d} Poisson\left(\frac{\lambda}{\mu}\right)$

Q/ If system starts empty ($X_0 = 0$), what's the 1st time we'll return to idle state?

$$\mathbb{E}[T_0 \mid X_0 = 0] = \frac{1}{q_0 \, \pi(0)} = \frac{e^{\lambda/\mu}}{\lambda}$$